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A GENERAL NON-CIRCULAR DUCT CONVECTIVE HEAT-TRANSFER PROBLEM FOR LIQUIDS AND GASES

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Abstract—The present paper is a theoretical study of the heat transfer by steady laminar forced convection in non-circular ducts. The region of cross-section of the duct under consideration is simply connected and bounded by an arbitrary non-circular closed curve. An arbitrary additional heat source distribution is assumed to be present within the fluid medium. The thermal boundary condition used is that the wall temperature varies linearly in the axial direction. Including viscous dissipation and work of compression in the thermal energy balance, the most general solution has been given in terms of integral formulas both for gases and liquids by using the technique of conformal mapping. General power series solution has been given for the case of gases only. For demonstration, the case of Cardioid duct with additional heat source distribution of constant intensity has been investigated numerically. Out of the mathematical work, only final results have been presented, and the description of methods have been deleted. To investigate the qualitative as well as the quantitative effects of viscous dissipation in the case of liquids, and those of viscous dissipation jointly with work of compression in the case of gases on heat transfer due to constant axial temperature gradient, is the principal object of the present study, and the emphasis also has been mainly given to this. It is found that the said effects are qualitatively remarkable, and usually significant quantitatively also under the condition of constant physical properties; which is a common simplification in a large number of heat-transfer studies. It is also concluded that if viscous dissipation and work of compression are significant in the heat-transfer problem of the present paper then the free convection effects thereby are insignificant.

NOMENCLATURE

a_n, constant quantities, appearing as coefficients in (37);
A, area of cross-section of the given duct;
B, arbitrary non-circular closed curve, as the boundary of each cross-section of the given duct;
C_p, specific heat at constant pressure and referred to weight;
C_1, parameter, $(1/\mu)(\mathrm{d}p/\mathrm{d}z')$;
C_2, parameter, $(\rho g C_p \tau/K)$;
C_3, heat source function or parameter, Q/K;
C_4, parameter, $C_1 C_2$;
D, domain of cross-section of the given duct, a simply connected finite region of boundary B;
D_e, equivalent hydraulic diameter, $4A/S$;
g, acceleration due to gravity;
h, heat-transfer coefficient;
i, fundamental imaginary quantity, $\sqrt{-1}$;
K, thermal conductivity;
l, physical length of the Cardioid cross-section defined by (51);
L, representative length of D;
Nu, Nusselt number;
p, pressure;
Pr, Prandtl number;
q, heat-transfer rate from wall to fluid;
Q, intensity of additional heat-source distribution;
r, radial coordinate in the physical plane, introduced in (51);
S, linear measure of B;

t, local temperature within the fluid medium;
t', time;
T, difference between local and wall temperatures, $t - t_w$;
u, local velocity of a fluid particle in z'-direction;
x, y, z', Cartesian co-ordinates, z'-axis is being parallel to the axis of the given duct;
z, complex variable, $x + iy$.

Greek symbols
α_n, known constant quantities, appearing as coefficients in (38);
β, coefficient of thermal expansion;
Γ, boundary of the unit circular domain in ζ-plane;
Δ, dilation of a fluid element;
ζ, complex variable in mathematical plane;
η, parameter, μ/K;
θ, argument of ζ;
μ, viscosity coefficient;
ξ, value of ζ at Γ;
ρ, density;
σ, vectorial angle in the physical plane, introduced in (51);
τ, constant axial temperature gradient, dt_w/dz'.

Operators
D/Dt', denotes total differentiation with respect to time;
∇^2, denotes two-dimensional Laplacian operator;
prime, denotes differentiation with respect to argument unless the contrary is specified;
bar, denotes conjugate complex, e.g. $\bar{z} = x - iy$;
$||$, denotes absolute value;
Re, denotes real part;
Im, denotes imaginary part.

Subscripts
i, condition at the entrance section;
m, mean value;
min, minimum value;
max, maximum value;
M, mixed mean value;
w, value at the wall.

Superscripts
(l), the case of liquids;
(g), the case of gases.

1. INTRODUCTION

A GROWING number of research workers have been engaged for some time in studying heat-transfer processes in non-circular duct flows. A man of technical science is interested because non-circular ducts are frequently encountered in almost all the branches of science and technology, e.g. from automobile radiators to nuclear power plants, experimental physics, biological sciences, etc. Circular tube is the simplest particular case of general non-circular tube, and is simplest in analysis also. Many difficulties are encountered when an attempt is made to approach non-circular cross-section. Therefore a mathematician is also interested because he has many of the mathematical tools to be useful in theoretical approach.

Moreover, there exists a basic difference between circular and non-circular cross-sections when the Neumann-type thermal condition is prescribed at the wall. In this situation the asymmetry of the non-circular cross-section gives rise to a circumferential temperature gradient, whereas this is not the case with circular cross-section. There is no such distinction between circular and non-circular ducts when thermal boundary condition is Dirichlet type.

The recent developments in the study of the general non-circular duct heat-transfer problem of Dirichlet type, which motivate the present study, are due to Tao [1–3]. Brief review of Tao's work, which is necessary, will be given in a later

Section. In the succeeding Section we propose to state our problem.

2. STATEMENT OF PROBLEM

Consider the laminar and steady flow of a Newtonian fluid in a sufficiently long straight duct of non-circular singly connected cross-section of constant area.

The cases under consideration are as follows:

(a) Arbitrary rectifiable closed contour as the boundary of each cross-section of the given duct.

(b) Arbitrary additional heat source distribution within the flow region.

(c) Linearly varying wall temperature in the axial direction as the thermal boundary condition.

The various simplifying assumptions to be made in the present study are as follows:

(i) The convection is regarded as forced convection by assuming that the free convection effects are comparatively negligible.

(ii) Fluid properties (viz. viscosity, thermal conductivity, etc.) are regarded as constant physical quantities.

(iii) The additional heat source distribution does not vary in the axial direction, and is given in terms of the remaining two space co-ordinates.

(iv) The hydrodynamic and thermal inlet lengths are insignificant as compared to the length of the regime of the fully established velocity and temperature profiles.

Thus, in view of the preceding simplifications, the region of flow to be discussed in the present study is that where velocity and temperature profiles have attained the fully developed conditions.

The essential purpose of the present paper is to take viscous dissipation and work of compression into consideration in the thermal energy balance of a moving fluid element, and then to give the general solution of the stated non-circular heat-transfer problem. The principal object of the present study then is to investigate the qualitative as well as the quantitative effects of the aforementioned two physical facts on the heat transfer in the stated forced convection due to the constant axial wall temperature gradient.

The present thermal boundary condition makes it relatively easier to obtain exact solutions. It is technically important also, as it is easily attainable in experiments and is reasonably satisfied, for example, in some nuclear reactor cooling problems and in some counter flow heat exchangers. The subject matter of taking additional heat sources into account is of current interest. Such a heat source distribution within the fluid medium is encountered, for instance, in certain nuclear reactor heat-transfer problems.

It may be said that the present problem is one of the most fundamental and important problems of heat transfer. It is useful in studying other heat-transfer problems of interest, e.g. mixed convection in which free convection effects may be treated as secondary effects; undeveloped regimes of velocity and temperature profiles; variable property problems, etc. The present problem is also useful, perhaps, in studying the case of non-Newtonian fluids. Here, we specially refer to those fluid flows, which obey Rivlin–Ericksen constitutive equation for second-order fluids; where non-Newtonian effects may be assumed to be secondary effects.

3. GOVERNING EQUATIONS

Under the aforegiven simplifying assumptions, the velocity field of any fluid in the present problem is governed by the following equation only:

$$\nabla^2 \mathring{u} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z'} = C_1. \tag{1}$$

Including the quantity $p\Delta$ in the thermal energy balance of a moving element of any fluid, one obtains the term:

$$(t/\rho)(\partial\rho/\partial t)_p(\mathrm{D}p/\mathrm{D}t')$$

in the energy equation, when C_p is employed as the operative specific heat. For liquids, this term can be dropped out of the energy equation in any case [4], since $(1/\rho)(\partial\rho/\partial t)_p$ is usually very small for liquids. For gases this quantity is not small, and usually comparable with dissipation function [4, 5]. According to the perfect gas law, one has $(t/\rho)(\partial\rho/\partial t)_p(\mathrm{D}p/\mathrm{D}t') = (-\mathrm{D}p/\mathrm{D}t')$. Thus we see that in the present problem, although the velocity field of a liquid and that of a gas remain the same, which is governed by equation (1), the temperature field in the case of liquids is different from that in the case of gases. Thus there should be two energy equations; one for liquids and the other for gases.

Before we give these equations, note that as a consequence of the given simplifying assumptions (i) to (iv) and the wall temperature varying linearly in the axial direction, the temperature of fluid (throughout this work, the word 'fluid' has been used for both liquids and gases) at any point varies linearly and at the same rate in the axial direction as the wall temperature does, i.e. $\partial t/\partial z' = \partial t_w/\partial z' =$ constant.

Assuming t_w to be constant in the peripheral direction, and letting $\mathrm{d}t_w/\mathrm{d}z' = \tau$ (a constant having dimensions of temperature gradient), the following are the energy equations of the present problem for liquids and gases respectively:

$$\rho g C_p \tau u = K\nabla^2 t^{(l)} + Q + \mu\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} \tag{2}$$

$$\rho g C_p \tau u = K\nabla^2 t^{(g)} + Q + \mu\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + u\frac{\mathrm{d}p}{\mathrm{d}z'}. \tag{3}$$

The mathematical statement for the boundary conditions to be used is as follows:

$$\left.\begin{aligned} &u(x, y) = 0 \quad \text{on} \quad B \\ &t^{(l)}(x, y, z') = t^{(g)}(x, y, z') = t_w(z') \\ &\qquad = t_i + \tau z' \quad \text{on} \quad B \end{aligned}\right\} \tag{4}$$

The superscripts (l) and (g) represent the cases of liquids and gases respectively. This convection is retained for other heat-transfer quantities which are going to occur later on. Any notation without the superscripts (l) and (g) is to be regarded as common to both liquids and gases.

The terms involving μ in equations (2) and (3) represent dissipation function. The term involving $\mathrm{d}p/\mathrm{d}z'$ in equation (3) is a consequence of the usually large thermodynamic pressure in gases [5], and is equal to $\mathrm{D}p/\mathrm{D}t'$ in the present case.

The pressure gradient $\mathrm{d}p/\mathrm{d}z'$ is constant, and can be determined either experimentally or from

$$\frac{\mathrm{d}p}{\mathrm{d}z'} = K_1 \mu u_m / L^2. \tag{5}$$

Here, K_1 is a negative real number; is different for different B; and is given by

$$K_1 = A/[\int_D (u/C_1 L^2)\,\mathrm{d}A]. \tag{6}$$

The boundary B of the simply connected region D is any closed contour. Furthermore, the heat source intensity Q is any arbitrarily prescribed function of x and y.

4. NECESSARY REVIEW OF THE WORKS OF TAO AND MADEJSKI

Neglecting the terms with μ as coefficient in equation (2), and neglecting the same terms and the term with $\mathrm{d}p/\mathrm{d}z'$ as coefficient in equation (3), the present system of governing equations reduces to that of Tao [1–3]. The mathematical formulation of this reduced system presented by Tao is somewhat unpleasnat and not simplest from a mathematics viewpoint. Tao unnecessarily reduces the evaluation of the temperature profile to the evaluation of two temperature profiles. One accounts for τ, and the other accounts for Q. The former was obtained as the result of solving biharmonic equation, and the latter was obtained as the result of solving Laplacian equation. However, the evaluation of the actual temperature profile does not require its partitioning and the involution of biharmonic equation in any case. This will be shown in a later section. Also, Tao has given the details of the complex variable methods of solving biharmonic and Laplacian equations. However, these methods and the related basic mathematical concepts can be found in [6–9].

The aims of Tao's three papers [1–3] are as follows: The purpose of [1] is to show that there exists a class of non-circular ducts for which the solution can be obtained directly from the equations of B. The purpose of [2] is to show that the solution can be obtained by means of the technique of conformal transformation when B is an arbitrary non-circular closed contour. And the purpose of [3] is to show the unreliability of the commonly used technique of equivalent circular duct.

The basic equations of Madejski's solution (i.e. the solution given by (8) in [10]) are obtained if we set $Q = \tau = 0$ in equations (3) and (4) of the present case. The method of obtaining the said solution in [10] is not mathematically rigorous, and not applicable in the cases of $Q \neq 0$ and $\tau \neq 0$. However, the purpose of [10] is to examine the effects of the inclusion of work of compression in the thermal energy balance on the temperature profile in the case of constant material properties. The cases of circular and flat conduits have been analysed completely.

In the author's knowledge, the other work, where the work of compression has been taken into consideration in the case of constant property flow, is due to Riley [11]. He has studied the heat-transfer problem of converging flow between non-parallel plane walls.

5. AMOUNT OF VISCOUS DISSIPATION AND COMPRESSION WORK IN THE PRESENT HEAT-TRANSFER PROBLEM

The omission of viscous dissipation in the thermal energy balance for any real fluid, and the omission of compression work in the thermal energy balance for any real gas are unrealistic from the physics of fluids. Thus Tao's works [1–3] are approximate studies of the stated heat-transfer problem, and do not show any distinction between liquids and gases. On the other hand Madejski's work [10] is not approximate, and marks the distinction between the heat transfer in liquids and that in gases. But the study in [10] is comparatively simpler and may be regarded as a particular case of the present one.

The DF (abbreviation for "dissipation function") and hence the TP (abbreviation for "total time derivative of pressure") may in general be ignored if and only if the temperature differences in the flow field are chiefly due to applied heating (e.g. the solid may be maintained at a certain temperature distribution [4]). This is true even in the case of non-solenoidal motion, since by means of applied heating, one can create non-solenoidality [12], and also mixed convection, turbulence, etc.

However, in the following we shall see that, in reasonably satisfying the assumptions (i, ii and iv) of the given assumptions (stated in Section 2; which have been made in Tao [1–3] also), the amount of DF in a liquid flow and the amounts of DF and TP in a gas flow come out to be significant.

Let L be the representative length, $|C_1|L^2$ be the representative velocity, and $|\tau|L$ be the representative temperature. Then the representative estimate (i.e. a relative measure with respect to heat of convection) for the amount of DF in a liquid flow, or that for the amount of DF or TP or DF + TP in a gas flow, for the present forced convection, from "dynamical similarity principle", is given by

$$|\eta'| = \frac{|\mathrm{d}p/\mathrm{d}z'|}{\rho g C_p |\tau|} = \frac{|K_1| \mu u_i}{\rho g C_p L^2 |\tau|}. \tag{7}$$

Here use of (5) has been made, and the mean velocity has been replaced by the entrance velocity; since they are equal in the present case.

In general, the effects of TP and hence those of DF in forced convection, under Dirichlet-type thermal boundary condition, are considerable if they are appreciably comparable with the amount of heat consumption per unit time to look forward to changing the temperature of a fluid element. This is certainly satisfied when the driving forces (i.e. pressure forces) are large or when the energy of motion and its total time rate of change are large. Mathematically speaking, for the present problem, this is nothing but to say that DF in the case of liquids, and DF and TP in the case of gases, are considerable if $|\eta'|$ bears significant value.

Before discussing the magnitude of the dimensionless number η', let us first obtain the relative amount of the free convection in the present forced convection problem, since this will be seen entangled with the desired discussion. Because of $\tau \neq 0$, a certain amount (however small) of free convection may be present in the present problem. Because of this free convection, regarded as the secondary convection imposed on the primary forced convection in the present case, the flow pattern and hence the temperature field are being altered. Using the same aforementioned representative quantities, we obtain the following dimensionless group under the "similarity analysis" to give the representative estimate of the relative amount of the free convection in the present forced convection

$$|\eta''| = \frac{g\beta|\tau|}{C_1^2 L^2} = \frac{g\beta|\tau| L^2}{K_1^2 u_i^2}. \tag{8}$$

Like η', η'' also holds good for both liquids and gases.

Note that both η' and η'' involve some of the material properties, the representative flow and thermal conditions, and the representative length of the system. Therefore, each of them is a complete parameter.

It is known from various studies and [4, 13, 14] that the hydrodynamic entry length is directly proportional to the square of L. Furthermore, it can be concluded from the available literature that the temperature distribution due to $\tau(\neq 0)$ over a cross-section is also proportional to L^2. Therefore, in order to satisfy the assumptions (ii) and (iv) reasonably, L must be sufficiently small.

In order that a material element has suffered negligible alteration of a physical property in its motion over a reasonable distance in the axial direction; and also in order that a physical property is reasonably constant throughout a cross-section, the magnitude of τ must be small.

Thus, in order to satisfy that the fluid temperature varies linearly and at the same rate in the axial direction as the wall temperature does, and keeping in view the temperature differences over a cross-section due to $Q(\neq 0)$, only low magnitudes of τ and L are permissible.

Given any fluid, there is a physical property which varies considerably with temperature. For instance, viscosity varies very rapidly with temperature in the case of liquids. In the case of gases, density is an important function of temperature. If we look into the table for water [13], we find that the ratio of viscosity at 0°C to that at 25°C is 2.

Calculations show that in the case of the circular tube flow of water with $u_i = 10{\cdot}0$ cm/s and initial temperature 20°C, the hydrodynamic entry length is approximately 115·0 cm when the radius is 1·0 cm. The absolute difference between the temperature at the wall and that at the axis, on the basis of the calculations made for fully developed flow, is approximately 245 degC due to $|\tau| = 0{\cdot}1$ degC/cm. This example does not satisfy the conditions (i), (ii) and (iv).

All this shows that L and $|\tau|$ are sufficiently small in the problem in hand, and as a consequence we find that essentially $|\eta'|$ bears significant value; $|\eta''|$ bears insignificant value; and hence the Grashof number (which is directly proportional to $|\tau| L^4$ in the present case) bears too insignificant value. Thus the assumption (i) is automatically satisfied.

Now, with sufficiently small L we can assign a large value to u_i within the limits of the laminar flow. For instance, it has been seen [4] that the flow behaviour of water does not change with $u_i = 1250{\cdot}0$ cm/s in the circular tube of radius 0·005 cm.

Calculations of several examples of B show that the constant K_1 has a magnitude which is greater than unity. This again signifies $|\eta'|$ and insignifies $|\eta''|$.

Thus our conclusions with regards to DF, TP and free convection in the heat-transfer problem in hand are as follows:

If DF in the case of liquids and DF and TP in the case of gases are considerable then the free convection effects are negligible in both the cases. It may, however, be remarked that the range of the present forced convection with considerable free convection, and the range of the intermediate case (i.e. the case of negligible DF, TP and free convection) are quite narrow.

For moderate values of u_i, DF in a liquid flow and DF and TP in a gas flow are definitely appreciable. These are negligible at quite a low speed.

Since it has appeared that we are concerned with conduits of narrow bore, let us remark that a conduit of narrow bore is very important in experiments. The problem of non-circular duct is important, because non-circular cross-section is frequently encountered with narrow bores. In the case of a broad bore (e.g. circular cross-section of radius equal to 1·0 cm), all of the given assumptions, excluding (iii), may not hold good satisfactorily.

The magnitudes of η' which represent negligible DF in liquids and negligible DF and TP in gases, will be investigated in the last section.

6. MOST GENERAL SOLUTION IN TERMS OF INTEGRAL FORMULAS

Since the number of factors of heat transfer considered here is greater than that considered in Tao [1–3], the mathematical expression of any heat-transfer result of the present study will be longer than that of the corresponding result of [2]. However, we should not mind it, since the additional factors considered here are physical facts (viz. viscous dissipation and work of compression).

The system of equations of heat transfer for liquids, and that for gases cannot be solved by using the method of Madejski, but can be solved most easily by employing the methods of complex variables.

Along with certain additional manipulations, the present system of governing equations (both for gases and liquids) can be solved just like that in [2]. The mathematical formulation, which is simplest, and which avoids the partitioning of the temperature profile and does not require involution with biharmonic equation, is given in this section. We need not give any details, but outlines and then final results only.

Introduce complex variables z and $\bar{z}$ in equations (1–4). Then transform them by means of appropriate conformal map from the physical plane to another complex plane where the physical domain D is being mapped onto unit circular domain. Now reduce each of the momentum and energy equations to Laplacian form by making a change of variable. Since the momentum equation is uncoupled with the energy equation of any case (the case of liquids as well as the case of gases), it can be solved independently.

Let $|\zeta| \leqslant 1$, with boundary Γ, be the unit circular domain in ζ-plane which is being mapped onto D in the z-plane by means of the conformal map

$$z = \Omega(\zeta). \tag{9}$$

The velocity distribution (both for gases and liquids) is, therefore,

$$u = \frac{C_1}{4}\{\Omega(\zeta)\bar{\Omega}(\bar{\zeta}) + 2\,\mathrm{Re}\,[y(\zeta)]\} \tag{10}$$

where

$$y(\zeta) = -\frac{1}{2\pi i}\int_{\Gamma}\frac{\Omega(\xi)\,\bar{\Omega}(1/\xi)}{\xi - \zeta}\,\mathrm{d}\xi - \bar{y}(0) \tag{11}$$

Using (9) to (11), and introducing

$$C_3(z, \bar{z}) = C_3(x, y) = (1/K)\, Q(x, y) \tag{12}$$

$$M(\zeta, \bar{\zeta}) = C_3\{\Omega(\zeta), \bar{\Omega}(\bar{\zeta})\} \tag{13}$$

$$g(\zeta, \bar{\zeta}) = \tfrac{1}{4} \iint M(\zeta, \bar{\zeta})\, \Omega'(\zeta)\, \bar{\Omega}'(\bar{\zeta})\, \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta} \tag{14}$$

$$G(\xi) = [g(\zeta, \bar{\zeta})]_\Gamma \tag{15}$$

$$\Phi(\zeta) = \int y(\zeta)\, \Omega'(\zeta)\, d\zeta \tag{16}$$

the temperature field in the case of liquids is given by

$$T^{(l)} = \frac{C_4 - \eta C_1^2}{64}\, \Omega^2(\zeta)\, \bar{\Omega}^2(\bar{\zeta}) + \frac{C_4 + \eta C_1^2}{8} \operatorname{Re}\, [\overline{\Omega(\zeta)}\, \Phi(\zeta)] - g(\zeta, \bar{\zeta})$$

$$+ 2 \operatorname{Re}\, [\psi^{(l)}(\zeta)] - \frac{\eta C_1^2}{16} \{\Omega(\zeta)\, \bar{\Omega}(\bar{\zeta}) \,.\, 2 \operatorname{Re}\, [y(\zeta)] + y(\zeta)\, \bar{y}(\bar{\zeta})\} \tag{17}$$

$$\psi^{(l)}(\zeta) = \frac{1}{2\pi i} \Bigg[- \frac{C_4 - \eta C_1^2}{64} \int_\Gamma \frac{\Omega^2(\xi)\, \bar{\Omega}^2(1/\xi)}{\xi - \zeta}\, \mathrm{d}\xi - \frac{C_4 + \eta C_1^2}{16} \int_\Gamma \frac{\bar{\Omega}(1/\xi)\, \Phi(\xi) + \Omega(\xi)\,\overline{}\,(1/\xi)}{\xi - \zeta}\, \mathrm{d}\xi$$

$$+ \frac{\eta C_1^2}{16} \Bigg\{ \int_\Gamma \frac{\Omega(\xi)\, \bar{\Omega}(1/\xi)\, [y(\xi) - \bar{y}(1/\xi)]}{\xi - \zeta}\, \mathrm{d}\xi + \int_\Gamma \frac{y(\xi)\, \bar{y}(1/\xi)}{\xi - \zeta}\, \mathrm{d}\xi$$

$$+ \int_\Gamma \frac{G(\xi)}{\xi - \zeta}\, \mathrm{d}\xi \Bigg\} \Bigg] - \bar{\psi}^{(l)}(0) \tag{18}$$

and the temperature field in the case of gases is given by

$$T^{(g)} = \frac{C_4}{64}\, \Omega^2(\zeta)\, \bar{\Omega}^2(\bar{\zeta}) + \frac{C_4}{8} \operatorname{Re}\, [\overline{\Omega(\zeta)}\, \Phi(\zeta)] - g(\zeta, \bar{\zeta}) + 2 \operatorname{Re}\, [\psi^{(g)}(\zeta)]$$

$$- \frac{\eta C_1^2}{32} \{\Omega(\zeta)\, \bar{\Omega}(\bar{\zeta}) + 2 \operatorname{Re}\, [y(\zeta)]\}^2 \tag{19}$$

$$\psi^{(g)}(\zeta) = \frac{1}{2\pi i} \Bigg[- \frac{C_4}{64} \int_\Gamma \frac{\Omega^2(\xi)\, \bar{\Omega}^2(1/\xi)}{\xi - \zeta}\, \mathrm{d}\xi - \frac{C_4}{16} \int_\Gamma \frac{\bar{\Omega}(1/\xi)\, \Phi(\xi) + \Omega(\xi)\, \bar{\Phi}(1/\xi)}{\xi - \zeta}\, \mathrm{d}\xi$$

$$+ \int_\Gamma \frac{G(\xi)}{\xi - \zeta}\, \mathrm{d}\xi \Bigg] - \bar{\psi}^{(g)}(0). \tag{20}$$

As yet, the heat source intensity is being regarded as an arbitrary function of x and y. The most general form of the heat source function, which can be allowed in the mathematical analysis in terms of complex variables, is the analytic function. Thus regarding $Q(x, y)$ as an analytic function in D, the heat source function can be expressed as

$$C_3(z, \bar{z}) = \operatorname{Re}\, [w(z)] \tag{21}$$

in the z-plane, or can be expressed as

$$M(\zeta, \bar{\zeta}) = \operatorname{Re}\, [W(\zeta)] \tag{22}$$

in the ζ-plane. The functions $w(z)$ and $W(\zeta)$ are analytic functions, at least in D and $|\zeta| \leqslant 1$ respectively.

Therefore, we have

$$g(\zeta, \bar{\zeta}) = \bar{\Omega}(\bar{\zeta})\, \Delta(\zeta) + \Omega(\zeta)\, \bar{\Delta}(\bar{\zeta}) \tag{23}$$

where

$$\Delta(\zeta) = \tfrac{1}{8} \int W(\zeta)\, \Omega'(\zeta)\, \mathrm{d}\zeta. \tag{24}$$

Now, the last but one term on the right-hand side of (18) or (20) can be set *a priori*.

From above, the case of constant heat source distribution is deduced as

$$g(\zeta, \bar{\zeta}) = \frac{C_3}{4}\, \Omega(\zeta)\, \bar{\Omega}(\bar{\zeta}). \tag{25}$$

We shall be dealing with the case of heat source function described by (22) in this and the succeeding Section only. Then, in remaining Sections, we shall deal with the particular cases of zero and constant heat source intensities.

For the present problem, the heat-transfer quantities of main interest are mean temperature T_m, mixed mean temperature T_M, heat-transfer rate q, heat-transfer coefficient h and Nusselt number Nu.

The first three of these quantities in the case of liquids are given as follows:

$$\begin{aligned} T_m^{(l)} = \frac{1}{A} \operatorname{Im} \Bigg[& \frac{C_4 - \eta C_1^2}{384} \int_\Gamma \Omega^2(\xi)\, \bar{\Omega}^3(1/\xi)\, \Omega'(\xi)\, \mathrm{d}\xi + \frac{C_4 + \eta C_1^2}{32} \int_\Gamma \Phi(\xi)\, \bar{\Omega}^2(1/\xi)\, \Omega'(\xi)\, \mathrm{d}\xi \\ & - \tfrac{1}{2} \int_\Gamma \Delta(\xi)\, \bar{\Omega}^2(1/\xi)\, \Omega'(\xi)\, \mathrm{d}\xi + \int_\Gamma \psi^{(l)}(\xi)\, \bar{\Omega}(1/\xi)\, \Omega'(\xi)\, \mathrm{d}\xi - \frac{\eta C_1^2}{32} \int_\Gamma y(\xi)\, \bar{\Phi}(1/\xi)\, \Omega'(\xi)\, \mathrm{d}\xi \\ & - \frac{\eta C_1^2}{32} \int_\Gamma \Omega(\xi)\, y(\xi)\, \bar{\Omega}^2(1/\xi)\, \Omega'(\xi)\, \mathrm{d}\xi \Bigg] \end{aligned} \tag{26}$$

$$T_M^{(l)} = \frac{C_1}{4u_m A} \operatorname{Im} (J_1 + J_2 + J_3 + J_4) \tag{27}$$

$$\begin{aligned} J_1 = & \frac{C_4 - \eta C_1^2}{64} \left[\tfrac{1}{8} \int_\Gamma \Omega^3(\xi)\, \bar{\Omega}^4(1/\xi)\, \mathrm{d}\Omega(\xi) + \tfrac{1}{3} \int_\Gamma \Omega^2(\xi)\, \bar{\Omega}^3(1/\xi)\, y(\xi)\, \mathrm{d}\Omega(\xi) \right] \\ & + \frac{C_4 + \eta C_1^2}{16} \Bigg[\int_\Gamma y(\xi)\, \Omega(\xi)\, \bar{\varphi}(1/\xi)\, \mathrm{d}\Omega(\xi) + \tfrac{1}{3} \int_\Gamma \Phi(\xi)\, \Omega(\xi)\, \bar{\Omega}^3(1/\xi)\, \mathrm{d}\Omega(\xi) \\ & + \tfrac{1}{2} \int_\Gamma y(\xi)\, \Phi(\xi)\, \bar{\Omega}^2(1/\xi)\, \mathrm{d}\Omega(\xi) \Bigg] \end{aligned} \tag{27a}$$

$$\begin{aligned} J_2 = - \frac{\eta C_1^2}{16} \Bigg[& \int_\Gamma y^2(\xi)\, \bar{\Phi}(1/\xi)\, \mathrm{d}\Omega(\xi) + \tfrac{3}{2} \int_\Gamma \Omega(\xi)\, y(\xi)\, \{\bar{\Omega}(1/\xi)\, \bar{\Phi}(1/\xi) - \bar{\varphi}(1/\xi)\}\, \mathrm{d}\Omega(\xi) \\ & + \tfrac{1}{3} \int_\Gamma \Omega^2(\xi)\, \bar{\Omega}^3(1/\xi)\, y(\xi)\, \mathrm{d}\Omega(\xi) + \tfrac{1}{2} \int_\Gamma \bar{\Omega}^2(1/\xi)\, \Omega(\xi)\, y^2(\xi)\, \mathrm{d}\Omega(\xi) \Bigg] \end{aligned} \tag{27b}$$

$$J_3 = \tfrac{1}{2}\int_\Gamma \Omega(\xi)\,\psi^{(l)}(\xi)\,\bar{\Omega}^2(1/\xi)\,d\Omega(\xi) + \int_\Gamma y(\xi)\,\psi^{(l)}(\xi)\,\bar{\Omega}(1/\xi)\,d\Omega(\xi) + \int_\Gamma \psi^{(l)}(\xi)\,\bar{\Phi}(1/\xi)\,d\Omega(\xi) \tag{27c}$$

$$J_4 = -\Big[\tfrac{1}{3}\int_\Gamma \Omega(\xi)\,\bar{\Omega}^3(1/\xi)\,\Delta(\xi)\,d\Omega(\xi) + \tfrac{1}{2}\int_\Gamma \Delta(\xi)\,y(\xi)\,\bar{\Omega}^2(1/\xi)\,d\Omega(\xi) + \int_\Gamma \Omega(\xi)\,y(\xi)\,\bar{\varphi}_1(1/\xi)\,d\Omega(\xi)\Big] \tag{27d}$$

$$\frac{q^{(l)}}{K} = C_2 u_m A - 4\,\mathrm{Im}\left\{\int_\Gamma \bar{\Delta}(1/\xi)\,\Omega'(\xi)\,d\xi\right\} - \frac{\eta C_1^2}{4}\,\mathrm{Im}\left\{\tfrac{1}{4}\int_\Gamma \Omega(\xi)\,\bar{\Omega}^2(1/\xi)\,\Omega'(\xi)\,d\xi + \tfrac{1}{2}\int_\Gamma y'(\xi)\,\bar{y}(1/\xi)\,d\xi + \int_\Gamma \bar{\Omega}(1/\xi)\,\bar{y}(1/\xi)\,\Omega'(\xi)\,d\xi - \int_\Gamma \bar{\Phi}(1/\xi)\,\Omega'(\xi)\,d\xi\right\}. \tag{28}$$

In deriving (26) to (28), we made use of the Complex Stokes Theorem [7]. Expecially, the Gauss Theorem was used in order to obtain the heat-transfer rate. The definition of T_M is

$$T_M = (1/u_m A)\int_D uT\,dA.$$

The quantities A, u_m, etc. are given below

$$A = \frac{1}{2i}\int_\Gamma \bar{\Omega}(1/\xi)\,\Omega'(\xi)\,d\xi \tag{29}$$

$$u_m = \frac{C_1}{4A}\,\mathrm{Im}\left\{\tfrac{1}{4}\int_\Gamma \Omega(\xi)\,\bar{\Omega}^2(1/\xi)\,\Omega'(\xi)\,d\xi + \int_\Gamma \bar{\Phi}(1/\xi)\,\Omega'(\xi)\,d\xi\right\} \tag{30}$$

$$\varphi(\zeta) = \int \Phi(\zeta)\,\Omega'(\zeta)\,d\zeta$$
$$\varphi_1(\zeta) = \int \Delta(\zeta)\,\Omega'(\zeta)\,d\zeta. \tag{31}$$

Similarly, using the Complex Stokes Theorem, the Gauss Theorem in making the energy balance over D, and (29) to (31), the same quantities (namely T_m, T_M and q) for the case of gases were obtained as

$$T_m^{(g)} = \frac{1}{A}\,\mathrm{Im}\Bigg[\frac{C_4}{384}\int_\Gamma \Omega^2(\xi)\,\bar{\Omega}^3(1/\xi)\,\Omega'(\xi)\,d\xi + \frac{C_4}{32}\int_\Gamma \Phi(\xi)\,\bar{\Omega}^2(1/\xi)\,\Omega'(\xi)\,d\xi + \int_\Gamma \psi^{(g)}(\xi)\,\bar{\Omega}(1/\xi)\,\Omega'(\xi)\,d\xi - \tfrac{1}{2}\int_\Gamma \Delta(\xi)\,\bar{\Omega}^2(1/\xi)\,\Omega'(\xi)\,d\xi - \frac{\eta C_1^2}{32}\left\{\tfrac{1}{6}\int_\Gamma \Omega^2(\xi)\,\bar{\Omega}^3(1/\xi)\,\Omega'(\xi)\,d\xi + \int_\Gamma \Omega(\xi)\,y(\xi)\,\bar{\Omega}^2(1/\xi)\,\Omega'(\xi)\,d\xi + \int_\Gamma y(\xi)\,\bar{\Phi}(1/\xi)\,\Omega'(\xi)\,d\xi + \int_\Gamma y^2(\xi)\,\bar{\Omega}(1/\xi)\,\Omega'(\xi)\,d\xi\right\}\Bigg] \tag{32}$$

$$T_M^{(g)} = \frac{C_1}{4u_m A}\,\mathrm{Im}\,(J_5 + J_6 + J_7 + J_8) \tag{33}$$

$$J_5 = \frac{C_4}{64}\Bigg[\tfrac{1}{8}\int_\Gamma \Omega^3(\xi)\,\bar{\Omega}^4(1/\xi)\,d\Omega(\xi) + \tfrac{1}{3}\int_\Gamma \Omega^2(\xi)\,\bar{\Omega}^3(1/\xi)\,y(\xi)\,d\Omega(\xi) + \tfrac{4}{3}\int_\Gamma \Phi(\xi)\,\Omega(\xi)\,\bar{\Omega}^3(1/\xi)\,d\Omega(\xi) + 2\int_\Gamma y(\xi)\,\bar{\Omega}^2(1/\xi)\,\Phi(\xi)\,d\Omega(\xi) + 4\int_\Gamma y(\xi)\,\Omega(\xi)\,\bar{\varphi}(1/\xi)\,d\Omega(\xi)\Bigg] \tag{33a}$$

$$J_6 = -\frac{\eta C_1^2}{32}\Bigg[\frac{1}{8}\int_\Gamma \Omega^3(\xi)\,\bar{\Omega}^4(1/\xi)\,d\Omega(\xi) + \int_\Gamma \Omega^2(\xi)\,\bar{\Omega}^3(1/\xi)\,y(\xi)\,d\Omega(\xi)$$
$$+ \frac{3}{2}\int_\Gamma \bar{\Omega}^2(1/\xi)\,\Omega(\xi)\,y^2(\xi)\,d\Omega(\xi) + 3\int_\Gamma y^2(\xi)\,\bar{\Phi}(1/\xi)\,d\Omega(\xi)$$
$$+ 3\int_\Gamma \Omega(\xi)\,y(\xi)\,\{\bar{\Omega}(1/\xi)\,\bar{\Phi}(1/\xi) - \bar{\varphi}(1/\xi)\}\,d\Omega(\xi) + \int_\Gamma y^3(\xi)\,\bar{\Omega}(1/\xi)\,d\Omega(\xi)\Bigg] \tag{33b}$$

$$J_7 = \frac{1}{2}\int_\Gamma \Omega(\xi)\,\psi^{(g)}(\xi)\,\bar{\Omega}^2(1/\xi)\,d\Omega(\xi) + \int_\Gamma y(\xi)\,\psi^{(g)}(\xi)\,\bar{\Omega}(1/\xi)\,d\Omega(\xi) + \int_\Gamma \psi^{(g)}(\xi)\,\bar{\Phi}(1/\xi)\,d\Omega(\xi) \tag{33c}$$

$$J_8 = J_4 \text{ see (27d)} \tag{33d}$$

$$\frac{q^{(g)}}{K} = AC_2u_m - 4\,\mathrm{Im}\left\{\int_\Gamma \bar{\Delta}(1/\xi)\,d\Omega(\xi)\right\}. \tag{34}$$

Now, the heat-transfer rate and the Nusselt number based on mixed mean temperature can be obtained both for gases and liquids from

$$h = -\,q/ST_M, \qquad Nu = hDe/K \tag{35}$$

where

$$S = \int_{-\pi}^{\pi} |\Omega'(\xi)|\,d\theta, \qquad De = 4A/S. \tag{36}$$

7. GENERAL POWER SERIES SOLUTION

It may be recognized that the evaluation of the various integrals in the ζ-plane, occurring in the preceding Section, depends on the nature of the mapping function (9). The purpose of this Section is to give the general series solution such that the solution for any given B and Q (i.e. for given $\Omega(\zeta)$ and $W(\zeta)$ respectively) is deducible directly from it.

It is known that every singly connected domain can be mapped onto the unit circular domain [15] in some different appropriate complex plane, and the mapping function can be obtained possibly [6] either in the exact or approximate form.

Furthermore [6], any given conformal map can be written in the form a power series. Let us, therefore, write

$$z = \Omega(\zeta) = \sum_0 a_n\zeta^n. \tag{37}$$

In general, (37) may be an infinite series and the constant coefficients a_n may be complex number. Since $W(\zeta)$ is an analytic function, $\Delta(\zeta)$ can be written also as

$$\Delta(\zeta) = \sum_0 \alpha_n\zeta^n. \tag{38}$$

Using (37), the quantities which remain the same both for gases and liquids were obtained in the series form as follows

$$u = \frac{C_1}{4}\left[\sum_0 a_n\zeta^n \sum_0 \bar{a}_n\bar{\zeta}^n - \left(\sum_0 D_n\zeta^n + \sum_0 \bar{D}_n\bar{\zeta}^n\right)\right] \tag{39}$$

$$u_m = \frac{C_1}{8\beta_0}\left[\sum_0 b_r\beta_{-r} + \sum_1 \bar{b}_r\beta_r - 4\sum_1 ra_r\bar{A}_r\right] \tag{40}$$

$$A = \pi \sum_0 ra_r\bar{a}_r$$

where

$$b_n = \sum_0 a_{n+r}\bar{a}_r, \qquad D_0 = \tfrac{1}{2}b_0; \qquad D_n = b_n \,(\text{when } n \geqslant 1)$$

$$\beta_n = \sum_0 (n+r)a_{n+r}\bar{a}_r; \qquad \beta_{-n} = \sum_0 r\bar{a}_{n+r}a_r \tag{42}$$

Now, using (37) to (42), we have for the case of gases

$$\begin{aligned} T^{(g)} = {} & \frac{C_4}{64}\sum_0 g_n\zeta^n \sum_0 \bar{g}_n\bar{\zeta}^n - \frac{C_4}{16}\left(\sum_0 \bar{a}_n\bar{\zeta}^n \sum_0 A_n\zeta^n + \sum_0 a_n\zeta^n \sum_0 \bar{A}_n\bar{\zeta}^n\right) \\ & + \sum_0 B_n\zeta^n + \sum_0 \bar{B}_n\bar{\zeta}^n - \left(\sum_0 \bar{a}_n\bar{\zeta}^n \sum_0 \alpha_n\zeta^n + \sum_0 a_n\zeta^n \sum_0 \bar{\alpha}_n\bar{\zeta}^n\right) \\ & - \frac{\eta C_1^2}{32}\left[\sum_0 a_n\zeta^n \sum_0 \bar{a}_n\bar{\zeta}^n - \left(\sum_0 D_n\zeta^n + \sum_0 \bar{D}_n\bar{\zeta}^n\right)\right]^2 \end{aligned} \tag{43}$$

$$\begin{aligned} T_m^{(g)} = {} & \frac{1}{\beta_0}\left[\frac{C_4}{192}\left(\sum_0 d_r\beta^-{}_r + \sum_1 \bar{d}_r\beta_r\right) - \frac{C_4}{16}\left(\sum_0 e_r\beta_{-r}\right) + \sum_1 e_r\beta_r\right) - \left(\sum_0 f_r\beta_{-r} + \sum_1 f_{-r}\beta_r\right) \\ & + 2\left(\sum_0 B_r\beta_{-r}\right) - \frac{\eta C_1^2}{16}\left\{\tfrac{1}{6}\left(\sum_0 d_r\beta_{-r} + \sum_1 \bar{d}_r\beta_r\right) - \left(\sum_0 (F_r - b_0D_r)\beta_{-r}\right.\right. \\ & \left.\left. + \sum_0 h_r\beta_{-r} + \sum_1 h_{-r}\beta_r\right) + \sum_0 G_r\bar{A}_r + \sum_0 H_r\beta_{-r}\right\}\Bigg] \end{aligned} \tag{44}$$

$$q^{(g)} = AKC_2u_m - 8\pi K \sum_0 ra_r\bar{\alpha}_r \tag{45}$$

where

$$A_n = (1/n)A'_n \quad (\text{when } n \geqslant 1); \qquad A_0 = A'_0 = 0; \qquad A'_n = \sum_0^n ra_rD_{n-r}, \qquad B_0 = \tfrac{1}{2}B'_0$$

$$B_n = B'_n \quad (\text{when } n \geqslant 1); \qquad B'_n = -\frac{C_4}{64}d_n + \frac{C_4}{16}(e_n + e_{-n}) + f_n + f_{-n}$$

$$d_n = \sum_0^n b_rb_{n-r} + 2\sum_1 b_{n+r}\bar{b}_r, \qquad e_n = \sum_0 A_{n+r}\bar{a}_r, \qquad e_n = \sum_0 \bar{a}_{n+r}A_r, \qquad f_n = \sum_0 \alpha_{n+r}\bar{a}_r$$

$$f_{-n} = \sum_0 \bar{a}_{n+r}\alpha_r, \qquad g_n = \sum_0^n a_ra_{n-r}, \qquad h_n = \sum_0 D_{n+r}\bar{b}_r; \qquad h_{-n} = \sum_0 \bar{b}_{n+r}D_r$$

$$F_n = \sum_0^n D_rb_{n-r}, \qquad G_n = \sum_0^n ra_rD_{n-r}, \qquad H_n = \sum_0^n D_rD_{n-r}.$$

Similarly, mixed-mean temperature and hence heat-transfer coefficient and Nusselt number can be calculated in the form of power series. But this should not be done here, because a large number of additional notations would be introduced. Also, for the same reason we do not propose to produce the heat-transfer quantities of the case of liquids in the power series form.

In many cases (e.g. nearly circular ducts [15], rectangular ducts, etc.) where mapping function does not exist in closed form, but in the form of an infinite series, the mathematical results in the form of general power series are useful. Applicability of such results becomes important when the given mapping function can be approximated by a polynomial, such as

$$z = \Omega_N(\zeta) = \sum_0^N a_r \zeta^r$$

in the sense that the polynominal $z = \Omega_N(\zeta)$ maps the circular region $|\zeta| \leqslant 1$ onto some domain D_N in z-plane which can be made to approach the true domain D as closely as desired. In many cases, only first four or five terms of the series expression of $\Omega(\zeta)$ give a good approximation to D [6].

8. FURTHER RESULTS OF INTEREST WHEN THE WALL TEMPERATURE IS CONSTANT AND THE ADDITIONAL HEAT SOURCES ARE ABSENT

When the wall is adiabatic, and the temperature does not vary in the axial direction, then the following holds good for the channel-laminar fully developed flows:

$$Au_m\left(C_p t_i + \frac{u_m^2}{2g}\right) = \int_D u\left(C_p t + \frac{u^2}{2g}\right) \mathrm{d}A. \tag{46}$$

This, in the present study, holds good for the case of gases if and only if $\tau = Q = 0$. In this case, the temperature field for gases reduces to

$$t^{(g)} = t_w - \frac{P_r}{2gC_p} u^2. \tag{47}$$

This is the temperature field which has been earlier obtained by Madejski [10], and has been said equal to the temperature distribution in the vortex tube. However, the temperature distribution in the case of liquids is not given by (47). It is quite different from (47), and also the condition of adiabatic wall is not satisfied. Hence (46) will not hold good for liquids. Therefore Madejski's discussion of liquids is not relevant.

Using (46) and (47), the temperature change at the wall, i.e. $t_w - t_i$, and the temperature drop at the location of maximum speed, i.e. $t_i - t_{\min}^{(g)}$, are given by

$$\Delta t_w^+ = K_2 Pr - (K_2 - 1), \qquad \Delta t_{\min}^+ = (K_2 - 1) + Pr(K_3 - K_2) \tag{48}$$

where

$$\Delta t_w^+ = t_w^+ - t_i^+, \qquad \Delta t_{\min}^+ = t_i^+ - t_{\min}^{(g)+}, \tag{49}$$

$$t^+ = \frac{2gC_p t}{u_m^2}, \qquad K_2 = \frac{(u^3)_m}{u_m^3}, \qquad K_3 = \frac{u_{\max}^2}{u_m^2}. \tag{50}$$

Although K_2 and K_3 do not involve the characteristic length of D, nevertheless they are different for different ducts. Thus Δt_w^+ and $\Delta t_{\min}^+$ depend on Prandtl number as well as on the configuration of the given duct.

9. ILLUSTRATIVE EXAMPLE: CARDIOID DUCT

In order to apply previous derivations, and to illustrate the effects of viscous dissipation in the case of liquid and the effects of viscous dissipation and compression work in the case of gases, let us select the example of Cardioid cross-section. Let the equation of the boundary be

$$r = 2l(1 + \cos\sigma). \tag{51}$$

The unit circular domain, $|\zeta| \leqslant 1$, in the ζ-plane is being mapped onto Cardioid domain (defined by 51) in the z-plane by

$$z = \Omega(\zeta) = l(1 + \zeta)^2. \tag{52}$$

As regards heat source function, let us consider only the case of constant heat source intensity, i.e. $C_3(x, y) = C_3$ = a positive constant quantity.

Using

$$u = \tfrac{1}{4}C_1 l^2\{(1 + \zeta)^2(1 + \bar{\zeta})^2 - 2\,\mathrm{Re}\,(3 + 4\zeta + \zeta^2)\} \tag{53}$$

$$A = 6\pi l^2, S = 16l, U_m = -\tfrac{17}{24}C_1 l^2 \tag{54}$$

the heat-transfer quantities for the case of liquids are

$$\begin{aligned} T^{(l)} = {} & \frac{C_4 - \eta C_1^2}{64} l^4\{(1 + \zeta)^4(1 + \bar{\zeta})^4 - 2\,\mathrm{Re}\,(35 + 56\zeta + 28\zeta^2 + 8\zeta^3 + \zeta^4)\} \\ & + \frac{C_4 + \eta C_1^2}{48} l^4\,\mathrm{Re}\,\{(114 + 176\zeta + 85\zeta^2 + 26\zeta^2 + 26\zeta^3 + 3\zeta^4) - (1 + \bar{\zeta})^2(36\zeta + 42\zeta^2 \\ & + 20\zeta^3 + 3\zeta^4)\} + \frac{\eta C_1^2}{8} l^4\,\mathrm{Re}\,\{(1 + \zeta)^2(1 + \bar{\zeta})^2(3 + 4\zeta + \zeta^2) - (35 + 56\zeta + 28\zeta^2 + 8\zeta^3 \\ & + \zeta^4)\} + \frac{\eta C_1^2}{16} l^4\{2\,\mathrm{Re}\,(13 + 16\zeta + 3\zeta^2) - (3 + 4\zeta + \zeta^2)(3 + 4\bar{\zeta} + \bar{\zeta}^2)\} \\ & + \frac{C_3 l^2}{4}\{2\,\mathrm{Re}\,(3 + 4\zeta + \zeta^2) - (1 + \zeta)^2(1 + \bar{\zeta})^2\} \end{aligned} \tag{55}$$

$$T_m^{(l)} = C_4 l^4\left(\frac{97}{144} + \frac{17}{24}C_3' + \frac{97}{288}\eta'\right) \tag{56}$$

$$T_M^{(l)} = C_4 l^4\left(\frac{30503}{32640} + \frac{97}{102}C_3' + \frac{13865}{72640}\eta'\right) \tag{53}$$

$$q^{(l)} = -\frac{C_4 K A^2}{6\pi}\left(\frac{17}{24} + C_3' + \frac{17}{24}\eta'\right) \tag{58}$$

$$h^{(l)} = \frac{3\pi K}{8l}\left[\left(\frac{17}{24} + C_3' + \frac{17}{24}\eta'\right)\Big/\left(\frac{30503}{32640} + \frac{97}{102}C_3' + \frac{13865}{32640}\eta'\right)\right] \tag{59}$$

$$N_u^{(l)} = \frac{\pi^2\,765\,(17 + 24C_3' + 17\eta')}{30503 + 31040C_3' + 13865\eta'} \tag{60}$$

where

$$C_3' = C_3/C_4 l^2. \tag{61}$$

Similarly, using (53), (54) and (61), the corresponding results in the case of gases are

$$T^{(g)} = \frac{C_4 l^4}{64}\{(1 + \zeta)^4(1 + \bar{\zeta})^4 - 2\,\mathrm{Re}\,(35 + 56\zeta + 28\zeta^2 + 8\zeta^3 + \zeta^4)\}$$

$$+\frac{C_4 l^4}{48}\operatorname{Re}\{(114+176\zeta+85\zeta^2+26\zeta^3+3\zeta^4)-(1+\bar{\zeta})^2(36\zeta+42\zeta^2+20\zeta^3$$
$$+3\zeta^4)\}+\frac{C_3 l^2}{4}\{2\operatorname{Re}(3+4\zeta+\zeta^2)-(1+\zeta)^2(1+\bar{\zeta})^2\}$$
$$-\frac{\eta C_1^2 l^4}{32}\{(1+\zeta)^2(1+\bar{\zeta})^2-2\operatorname{Re}(3+4\zeta+\zeta^2)\}^2 \quad (62)$$

$$T_m^{(g)} = C_4 l^4\left(\frac{97}{144}+\frac{17}{24}C_3'-\frac{97}{288}\eta'\right) \quad (63)$$

$$T_M^{(g)} = C_4 l^4\left(\frac{30503}{32640}+\frac{97}{102}C_3'-\frac{16638}{32640}\eta'\right) \quad (64)$$

$$q^{(g)} = -\frac{C_4 K A^2}{6\pi}\left(\frac{17}{24}+C_3'\right) \quad (65)$$

$$h^{(g)} = \frac{3\pi K}{8l}\left\{\left(\frac{17}{24}+C_3'\right)\bigg/\left(\frac{30503}{32640}+\frac{97}{102}C_3'-\frac{16638}{32640}\eta'\right)\right\} \quad (66)$$

$$N_u^{(g)} = \frac{\pi^2 765\,(17+24C_3')}{(30503+31040C_3'-16638\eta')}. \quad (67)$$

In the case of gases, the quantities Δt_w^+ and $\Delta t_{\min}^+$ of Section 8 are, for Cardioid duct,

$$\Delta t_w^+ = \frac{49914}{24565}Pr - \frac{25349}{24565},$$
$$\Delta t_{\min}^+ = 1{\cdot}031915 + 2{\cdot}056569\,Pr \quad (68)$$

In the absence of heat sources (i.e. $C_3 = 0$) if ε_1, ε_2, ε_3 and ε_4 be the error in per cent of heat-transfer rate, mean temperature, mixed-mean temperature and Nusselt number respectively due to the omission of DF in liquid flows and DF and TP in gas flows, then these are given as follows:

In the case of liquids, we have

$$\varepsilon_1^{(l)} = \frac{100\eta'}{1+\eta'};\qquad \varepsilon_2^{(l)} = \frac{100\eta'}{2+\eta'}$$
$$\varepsilon_3^{(l)} = \frac{100\eta'}{2{\cdot}2+\eta'};\qquad \varepsilon_4^{(l)} = \frac{600\eta'}{11(1+\eta')} \quad (69)$$

and in the case of gases

$$\varepsilon_1^{(g)} = 0;\qquad \varepsilon_2^{(g)} = \frac{-100\eta'}{2-\eta'}$$
$$\varepsilon_3^{(g)} = \frac{-600\eta'}{11-6\eta'};\qquad \varepsilon_4^{(g)} = \frac{600}{11}\eta'. \quad (70)$$

10. DISCUSSION AND CONCLUDING REMARKS

Looking into (68) we find that Δt_w^+ is negative, i.e. cooling effect at the wall, when $Pr < 0{\cdot}508$. Heating effect is expected when $Pr > 0{\cdot}508$. These results are different from the corresponding results of circular and flat conduits [10]. This confirms the dependence of Δt_w^+ and $\Delta t_{\min}^+$ on duct configuration. Tables based on experiments show that for air and many other gases under ordinary conditions of temperature and pressure the Prandtl number is greater than 0·508. It may thus be concluded here that in the problem of Section 8 which is valid for gases only the heating effect will usually be observed at the wall. In the case of air at 15°C, calculations with $u_i = 10$ m/s give $t_i - t_{\min}^{(g)} = 0{\cdot}073$°C and those with $u_i = 50$ m/s give $t_i - t_{\min}^{(g)} = 1{\cdot}825$°C. Thus in the channel laminar gas flow with entrance speed up to 10 m/s the temperature differences $t_i - t_{\min}^{(g)}$ and $t_w - t_i$ are quite small. Figure 1 shows the variations of Δt_w^+ and $\Delta t_{\min}^+$ with Prandtl number.

Before proceeding further to illustrate the effects of DF in the case of liquids and the combined effects of DF and TP in the case of gases, both qualitative and quantitative in each case, the following should be noted first. The heat source parameter C_3' and the parameter η' are both either positive or negative. The first case occurs when $\tau < 0$, and the other case occurs when $\tau > 0$.

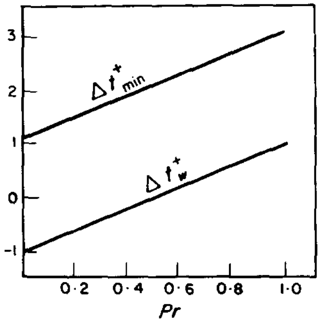

FIG. 1. Cardioid duct gas flow: temperature change at the wall and temperature drop at the line of maximum velocity vs. Prandtl number.

If we compare (28) and (34), we find that $q^{(g)}$ does not involve η'; but $q^{(l)}$ involves η'. We can confirm this in particular by comparing (58) and (65). This tells us that the combined effect of DF and TP on $q^{(g)}$ is zero. From this we can conclude that the contributions of DF and TP to $q^{(g)}$ are equal in magnitude, but opposite in direction. However, the contributions of DF and TP to heat-transfer rate have different magnitudes within the gaseous medium except at the location of maximum velocity where each is individually zero. Thus we find that the heat-transfer rate at the solid boundary only, in the case of gases, remains the same as that in the case of Tao; i.e. the case of negligible DF and TP. This statement is untrue for the case of liquids. Therefore one is required to discuss the effects of DF on $q^{(l)}$. Under crucial examination of (58) we find that the effect of DF is to diminish heat transfer when surface transfers heat to fluid (which is possible if and only if $\tau > 0$) and to magnify heat transfer when heat is transferred to surface (which is always true when $\tau < 0$ and also possible when $\tau > 0$). Certain more interesting points may be seen in Fig. 2, which gives the graphical variation of the dimensionless heat-transfer rate

$$q^{(l)'}(=q^{(l)}/C_4KA^2)$$

against the heat source number C_3' where η' is being regarded as parameter. The curve representing the case of zero dissipation (i.e. $\eta' = 0$) in Fig. 2 agrees with Fig. 1 of [2].

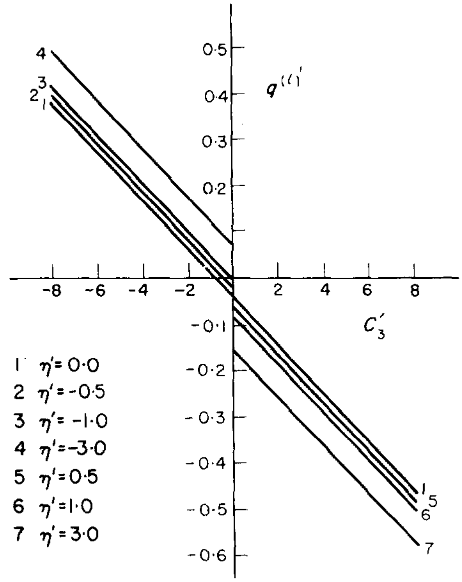

FIG. 2. Cardioid duct liquid flow: heat-transfer rate $q^{(l)'}$ vs. heat source number C_3' with η' as parameter.

Although we have seen that the combined effect of DF and TP on $q^{(g)}$ is zero, this is not the case with other relevant heat-transfer quantities of the case of gases. For instance, see (63), (64), (66) and (67).

From (63) and (64), we conclude that the combined effect of DF and TP is to inhibit the dimensionless mean temperature $T_m^{(g)}/C_4l^4$ as well as the dimensionless mixed-mean temperature $T_M^{(g)}/C_4l^4$ when $\tau < 0$, and to augment both when $\tau > 0$. On the other hand, from (56) and (57), we note that the effect of DF is to augment $T_m^{(l)}/C_4l^4$ and $T_M^{(l)}/C_4l^4$ when $\tau < 0$, and to diminish both when $\tau > 0$. Thus we see that the combined effects of DF and TP are opposite to the corresponding effects of DF in nature. This clearly shows that DF and TP produce effects which are opposite in nature, also that the intensity of the effects of TP is greater than that of the corresponding effects of DF.

Now, we come to the discussion of Nusselt number. The combined effects of DF and TP, in the case of gases, on the Nusselt number are shown in Fig. 3. The effects of DF, in the case of liquids, on the Nusselt number can be seen in Figs. 4 and 5. Figs. 3, 4 and 5 give a picture of the variation of the Nusselt number vs. C_3' when η' plays the role of independent parameter. For the case of gases we have shown both the cases: $\tau > 0$ and $\tau < 0$ in Fig. 3, but for the case of liquids we found it convenient to picture them separately; Fig. 4 represents the case $\tau > 0$ and Fig. 5 represents the case of $\tau < 0$.

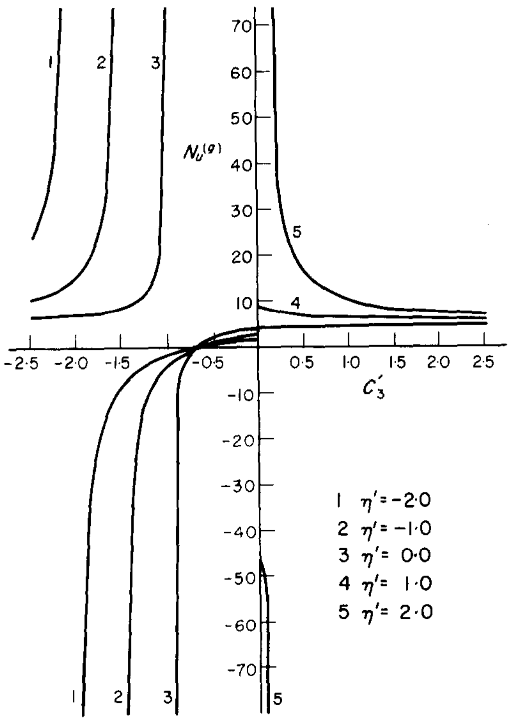

FIG. 3. Cardioid duct gas flow: Nusselt number $Nu^{(g)}$ vs. heat source number C_3' at various values of η'.

We note the following for $Nu^{(g)}$ and $Nu^{(l)}$ when $\tau > 0$:

(i) $Nu^{(g)}$ becomes zero at $C_3' = -17/24$ for all values of η'. $Nu^{(l)}$ becomes zero only when η' lies in a certain finite interval. To each value of C_3' lying in some different finite interval at which $Nu^{(l)}$ becomes zero, there corresponds one and only one value of η' in the said interval.

(ii) For each value of η', there exists a value of C_3' at which $Nu^{(g)}$ becomes infinite. But, this is not the case with $Nu^{(l)}$. $Nu^{(l)}$ becomes infinite when η' lies in a certain finite interval.

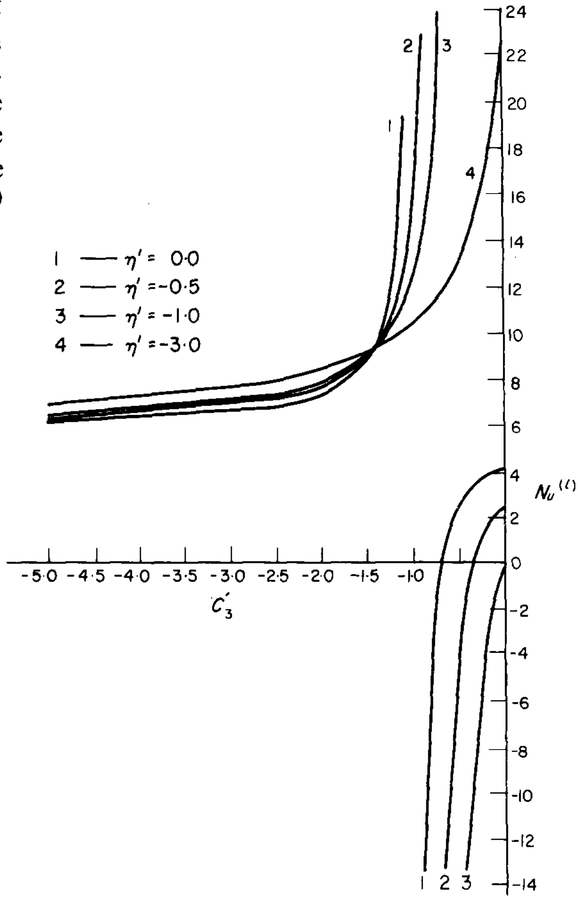

FIG. 4. Cardioid duct liquid flow (the case of $\tau > 0$): Nusselt number $Nu^{(l)}$ vs. heat source number C_3' at few values of η'.

(iii) In Fig. 4, we see that there exists a value of C_3' at which $Nu^{(l)}$ remains the same for all values of n'. Calculations show that this value of C_3' is nearly 1·451.

Further, our discussion of $Nu^{(l)}$ and $Nu^{(g)}$ when $\tau < 0$ is as follows:

(i) Neither $Nu^{(g)}$ nor $Nu^{(l)}$ becomes zero at any value of C_3' for all values of η'.

(ii) For all values of η', $Nu^{(l)}$ does not become infinite and remains finite at each value of C_3'. But the condition of $Nu^{(g)}$ is somewhat different: It remains finite and non-negative at each value of C_3' till $\eta' < 11/6$. Non-negative values and discontinuity of $Nu^{(g)}$ are observed only when $\eta' \geqslant 11/6$.

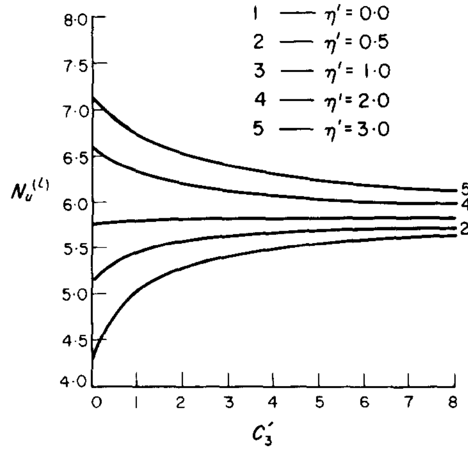

FIG. 5. Cardioid duct liquid flow (the case of $\tau < 0$): Nusselt number $Nu^{(l)}$ vs. heat source number C_3' at few values of η'.

(iii) There does not exist any value of C_3' at which either $Nu^{(l)}$ or $Nu^{(g)}$ becomes constant for all values of η'. However, we have a value of η' at which Nusselt number is same for all values of C_3'. For liquids this is $\eta' = 1{\cdot}04$ approximately, and for gases it is $\eta' = 0{\cdot}51$ approximately.

Several more interesting points may be noted from Figs. 3–5.

In summing up, one can say that the qualitative effects of DF are quite interesting and remarkable when $\tau > 0$. The combined qualitative effects of DF and TP are equally interesting and remarkable when $\tau > 0$. Although the effects of DF in the case of $\eta' > 0$ are not that interesting as they are in the case of $\eta' < 0$, but the combined effects of DF and TP in the case of $\eta' > 0$ are not less interesting as compared to those in the case of $\eta' < 0$; and rather the case of $\eta' > 0$ is more interesting than the case of $\eta' < 0$.

In order to visualize the combined quantitative effects of DF and TP in the case of gases and the quantitative effects of DF in the case of liquids, we present Figs. 6, 7 and 8. Figure 6 shows the picture of the variations of $\varepsilon_2^{(g)}$, $\varepsilon_3^{(g)}$ and $\varepsilon_4^{(g)}$ vs. η'. In this Figure, the axis of η' can be regarded as the graph of $\varepsilon_1^{(g)}$. Figures 7 and 8 represent the case of liquids. The variations of $\varepsilon_2^{(l)}$ and $\varepsilon_3^{(l)}$ vs. η' have been pictured in Fig. 7. And $\varepsilon_1^{(l)}$ and $\varepsilon_4^{(l)}$ have been pictured in Fig. 8.

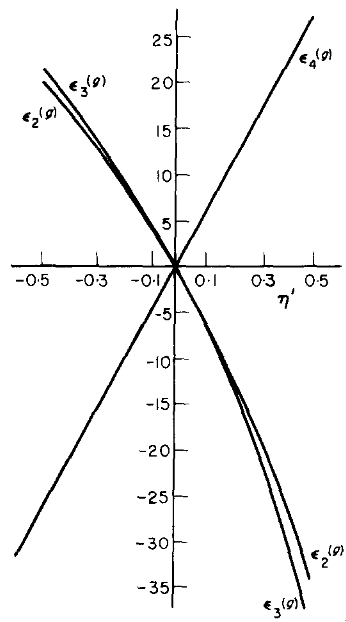

FIG. 6. Cardioid duct gas flow: variations of $\epsilon_2^{(g)}$, $\epsilon_3^{(g)}$ and $\epsilon_4^{(g)}$ with η'.

In these Figures, we see that the errors are significant even at small values of $|\eta'|$. In this connexion, furthermore, we look into Table 1.

We find that the quantitative effects of DF, and the combined quantitative effects of DF and TP may be insignificant when $|\eta'| \leqslant 0{\cdot}1$. However, they are considerable when $|\eta'| > 0{\cdot}1$. Note that at $\eta' = 1$ we have $\varepsilon_1^{(l)} = 50$, $\varepsilon_2^{(l)} = 33{\cdot}33$, $\varepsilon_3^{(l)} = 31{\cdot}25$, $\varepsilon_4^{(l)} = 27{\cdot}27$, $\varepsilon_2^{(g)} = -100{\cdot}0$, $\varepsilon_3^{(g)} = -120{\cdot}0$ and $\varepsilon_4^{(g)} = 54{\cdot}545$.

One may now ask: Do some physical examples exist which belong to the subject matter of the present study and where η' assume magnitudes such as $|\eta'| > 0{\cdot}1$? The answer is "yes", and the following examples are given.

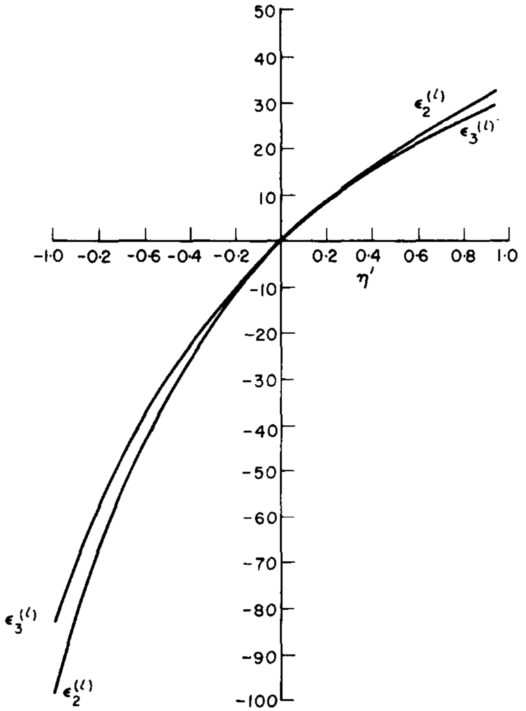

FIG. 7. Cardioid duct liquid flow: variations of $\epsilon_2^{(l)}$ and $\epsilon_3^{(l)}$ with η'.

Numerical example 1

Consider the flow of water through a duct of finite length, say 50 cm. Let $D_e = 0{\cdot}05$ cm, $u_i = 3 \times 10^2$ cm/s, $|\tau| = 0{\cdot}005$°C/cm and $t_i = 20$°C. The relevant physical properties of water in c.g.s. system at initial temperature 20°C are: $\mu = 1{\cdot}002 \times 10^{-2}$; $\rho = 0{\cdot}998$, $gC_p = 0{\cdot}999$.

Numerical example 2

Consider the flow of air through the duct of the previous example. Let $D_e = 0{\cdot}2$ cm, $u_i = 10^{-3}$ cm/s, $|\tau| = 0{\cdot}05$ degC/cm and $t_i = 15$°C. The relevant physical properties of air in c.g.s. system at initial temperature 15°C are: $\mu = 1{\cdot}8 \times 10^{-4}$, $\rho = 1{\cdot}203 \times 10^{-3}$, $gC_p = 0{\cdot}240$. The conversion factor of gC_p is $J = 4{\cdot}184 \times 10^7$.

For Cardioid cross-section, our calculations show that $|\eta'| > 0{\cdot}2$ in the first example and $|\eta'| > 0{\cdot}26$ in the second example. Obviously these are significant magnitudes of η'.

Performing additional calculations, one can verify that all the conditions, i.e. the assumptions of laminar flow, constant physical properties, negligible free convection, and negligible entry length, are sensibly satisfied in both the examples.

In the case of oils, e.g. typical aircraft engine oil, the magnitude of η' is usually significant even at low speeds, and is much larger than that in the case of water.

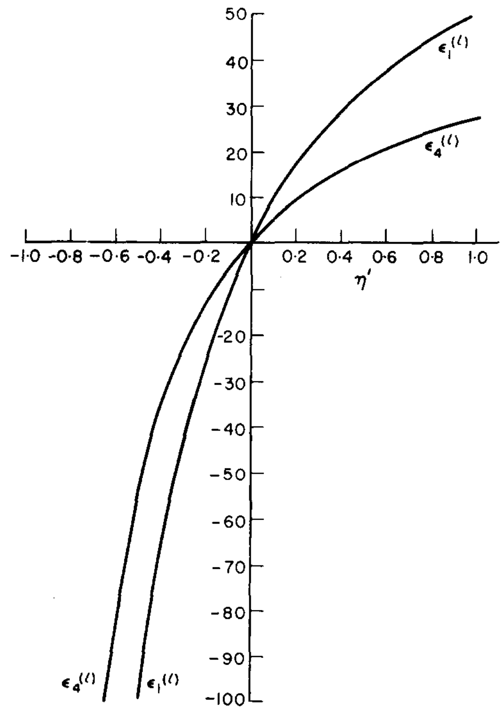

FIG. 8. Cardioid duct liquid flow: variations of $\epsilon_1^{(l)}$ and $\epsilon_4^{(l)}$ with η'.

Table 1

η'	$\epsilon_1^{(l)}$	$\epsilon_2^{(l)}$	$\epsilon_3^{(l)}$	$\epsilon_4^{(l)}$	$\epsilon_2^{(g)}$	$\epsilon_3^{(g)}$	$\epsilon_4^{(g)}$
0·1	9·0909	4·7619	4·3478	4·9586	−5·263	−5·769	5·45
−0·1	−11·1111	−5·2632	−4·7619	−6·0606	4·762	5·172	5·45

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Аннотация—Приводится теоретическое исследование теплообмена в установившемся ламинарном потоке при вынужденной конвекции в трубах некруглого сечения. Исследуемая область поперечного сечения трубы ограничена произвольной замкнутой кривой. Предполагется, что в потоке имеет место произвольное распределение дополнительного источника тепла. Используемое тепловое граничное условие состоит в том, что температура стенки линейно изменяется в аксиальном направлении. С учетом вязкой диссипации и работы сжатия в уравнении теплового баланса методом конформных преобразований получено наиболее общее решение в виде интегральных формул для газов и жидкостей. Общее решение степенного ряда дается только для газов. Для иллюстрации численно исследовался случай кардиоидной трубы с дополнительным тепловым источником постоянной мощности. Приводятся только окончательные результаты без математических выкладок. Основной целью данной статьи является исследование качественного и количественного влияния вязкой диссипации для жидкостей, а также этого же влияния с учетом работы сжатия для газов на тепообмен за счет постоянного аксиального температурного градиента. Найдено, что указанные эффекты качественно заметны и обычно количественно проявляются при постоянных физических свойствах, что является обычным упрощением в ряде исследований по теплообмену. Также сделан вывод, что если вязкая диссипация и работа сжатия в задаче о теплообмене, рассматриваемой в данной статье, значительны, то свободная конвекция будет незначительной.